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THE AVERAGING OF SYSTEMS WITH A HIERARCHY OF PHASE ROTATION SPEEDS†

A. V. PECHENEV

St Petersburg

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An investigation is carried out of a system of ordinary equations with rotating phases whose frequencies (rotation speeds) form a hierarchy in terms of powers of a small parameter. Attention is also devoted to a more complex system, whose right-hand sides also contain terms "with zero means" (averaged over trajectories of fast motions). The scheme proposed to deal with such systems involves the successive application of a standard procedure which isolates the "fastest" variables to within a certain accuracy in terms of the small parameter. The degree of correspondence between the solutions of exact equations and those of the equations thus averaged is determined over an asymptotically large time interval, during which the "slowest" variable increases by one order of magnitude.

SYSTEMS of equations of the type considered, with hierarchies of frequencies, arise both in treatments of the problem in [1, 2] and in the light of certain special features, emphasized in [3, 4], of the theory of simple resonance in essentially non-linear systems: when averaging procedures are applied to resonance trajectories, it turns out that the equations of the first approximation are Hamiltonian (as observed for a special case in [5]). As a consequence one obtains a "stratification" of motions in accordance with a hierarchy of speeds, more complicated than the traditional division into fast and slow motions.

The approach in question is effective because it achieves a reduction of the initial system at least to the same degree as in the non-resonant case. In the traditional treatment, however, "investigation of the resonant case always leads to averaged systems of higher dimensions" [6].

The averaging scheme proposed here must be justified as a single entity, because it is not obvious that individual "levels" of averaging may legitimately be applied over an asymptotically large time interval. In fact, averaging with respect to the "fastest" variable (called "partial" averaging in [4]) has been justified only over a time interval that is too short for the "slowest" variable to increase by one order of magnitude.

A sequence of averaging procedures has been justified as a single entity for the case of a hierarchy of two rotating phases [7]. The treatments in [3, 8] differ from that presented here.

1. STATEMENT OF THE PROBLEM

We consider a system of equations in which the frequencies (phase rotation speeds) form a hierarchy in terms of powers of a small positive parameter ε:

$$x^{\cdot} = \varepsilon^n X, \quad \alpha_{n-1}^{\cdot} = \varepsilon^{n-1} A_{n-1}, \ \ldots, \quad \alpha_0^{\cdot} = A_0 \tag{1.1}$$

where $X, A_n^{-1}, \ldots, A_0$ are functions of real arguments $x, \alpha_{n-1}, \ldots, \alpha_0, \varepsilon$, defined for $\varepsilon \in [0, \varepsilon_0]$ in some domain D of x space and 2π-periodic in the variables $\alpha_{n-1}, \ldots, \alpha_0$. The functions on the right of (1.1) are assumed to be n times differentiable with respect to x and i times differentiable with respect to α_i ($i = n-1, \ldots, 1$), uniformly with respect to x in D and ε in $[0, \varepsilon_0]$; they are also

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assumed to be measurable as functions of α_0. In addition, the functions A_i are bounded away from zero uniformly with respect to x in D and ε in $[0, \varepsilon_0]$.

We will construct a sequence of n changes of variables, each splitting off an equation for the next phase (in order of decreasing rotation frequencies). The right-hand sides of the remaining equations will continue to depend on the split-off phases only in small terms, of the order of ε^{m+1} for the slowest variable and $\varepsilon^{m-n+i+1}$ for the ith rotating phase, where $m \geqslant n$ is a number that ultimately determines the accuracy of the final averaged system.

For example, the kth change of variables $(k = 1, \ldots, n)$,

$$y = z + \varepsilon^{n-k+1} u_n, \quad \beta_{n-1} = \gamma_{n-1} + \varepsilon^{n-k} u_{n-1}, \quad \ldots, \quad \beta_k = \gamma_k + \varepsilon u_k \tag{1.2}$$

where u_i $(i = n, \ldots, k)$ are functions of $z, \gamma_{n-1}, \ldots, \gamma_{k-1}, \varepsilon$, uniformly bounded with respect to z in D and ε in $[0, \varepsilon_0]$, reduces the system of equations

$$\begin{aligned}
y^{\cdot} &= \varepsilon^n Y(y, \beta_{n-1}, \ldots, \beta_{k-1}, \varepsilon) + \varepsilon^{m+1} E_n \\
\beta_{n-1}^{\cdot} &= \varepsilon^{n-1} B_{n-1}(y, \beta_{n-1}, \ldots, \beta_{k-1}, \varepsilon) + \varepsilon^m E_{n-1} \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
\beta_k^{\cdot} &= \varepsilon^k B_k(y, \beta_{n-1}, \ldots, \beta_{k-1}, \varepsilon) + \varepsilon^{m-n+k+1} E_k \\
\beta_{k-1}^{\cdot} &= \varepsilon^{k-1} B_{k-1}(y, \beta_{n-1}, \ldots, \beta_{k-1}, \varepsilon) + \varepsilon^{m-n+k} E_{k-1}
\end{aligned}$$

to the equivalent form

$$\begin{aligned}
z^{\cdot} &= \varepsilon^n Z(z, \gamma_{n-1}, \ldots, \gamma_k, \varepsilon) + \varepsilon^{m+1} F_n \\
\gamma_{n-1}^{\cdot} &= \varepsilon^{n-1} \Gamma_{n-1}(z, \gamma_{n-1}, \ldots, \gamma_k, \varepsilon) + \varepsilon^m F_{n-1} \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
\gamma_k^{\cdot} &= \varepsilon^k \Gamma_k(z, \gamma_{n-1}, \ldots, \gamma_k, \varepsilon) + \varepsilon^{m-n+k+1} F_k \\
\beta_{k-1}^{\cdot} &= \varepsilon^{k-1} \Gamma_{k-1}(z, \gamma_{n-1}, \ldots, \gamma_k, \beta_{k-1}, \varepsilon) + \varepsilon^{m-n+k} F_{k-1}
\end{aligned}$$

by averaging over the β_{k-1} trajectory.

The definition of the *a priori* unknown functions $u_n, \ldots, u_k, Z, \Gamma_{n-1}, \ldots, \Gamma_{k-1}$ involves no difficulties if the right-hand sides of Eqs (1.1) are smooth functions of ε in the domain of interest. There is no need to specify the functions $F_n, \ldots, F_{k-1}$.

As to the equations whose left-hand sides are not affected by the change (1.2), their right-hand sides are expanded in Taylor series, being expressed, as a result, in terms of the newly introduced variables instead of the old ones.

After n changes of variables of type (1.2), system (1.1) is reduced to the equivalent system

$$\begin{aligned}
\xi^{\cdot} &= \varepsilon^n \Xi(\xi, \varepsilon) + \varepsilon^{m+1} G_n \\
\varphi_{n-1}^{\cdot} &= \varepsilon^{n-1} \Phi_{n-1}(\xi, \varphi_{n-1}, \varepsilon) + \varepsilon^m G_{n-1} \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
\varphi_0^{\cdot} &= \Phi_0(\xi, \varphi_{n-1}, \ldots, \varphi_0, \varepsilon) + \varepsilon^{n-m+1} G_0
\end{aligned} \tag{1.3}$$

Along with system (1.3), which we call the equations of the mth approximation, we will also consider what we call the truncated system of equations of the mth approximation:

$$\begin{aligned}
\eta^{\cdot} &= \varepsilon^n \Xi(\eta, \varepsilon) \\
\psi_{n-1}^{\cdot} &= \varepsilon^{n-1} \Phi_{n-1}(\eta, \psi_{n-1}, \varepsilon) \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
\psi_0^{\cdot} &= \Phi_0(\eta, \psi_{n-1}, \ldots, \psi_0, \varepsilon)
\end{aligned} \tag{1.4}$$

The question is, does the solution of system (1.4) yield a satisfactory approximation to that of the original system (1.1)?

2. JUSTIFICATION OF THE USE OF THE TRUNCATED EQUATIONS OF THE mth APPROXIMATION

Under the conditions stated in Sec. 1, the functions on the right of equations (1.4) are n times differentiable with respect to η and i times differentiable with respect to ψ_i, $i=n-1, \ldots, 0$ (referring, of course, to functions that depend on ψ_i), uniformly with respect to η in D and ε in $[0, \varepsilon_0]$; moreover, Φ_0 is a measurable function of ψ_0 and the functions Φ_i are bounded away from zero. In addition, the functions G_i on the right of Eqs (1.3) are bounded uniformly with respect to ξ in D and ε in $[0, \varepsilon_0]$.

Lemma. Under the above conditions, if the systems of equations (1.3) and (1.4) are integrated with the same initial data for corresponding variables over a time interval of the order of ε^{-n}, the following estimates will hold (on the added assumption that the ξ and η trajectories do not leave D):

$$|\xi-\eta| = O(\varepsilon^{m-n+1}) \tag{2.1}$$

$$|\varphi_i-\psi_i| = O(\varepsilon^{m-2n+i+1}), \quad i=n-1, \ldots, 0 \tag{2.2}$$

The proof of (2.1) involves comparing the first equations of systems (1.3) and (1.4), using the Lipschitz condition for the function Ξ and Gronwall's Lemma.

We shall prove only the estimate

$$|\varphi_{n-1}-\psi_{n-1}| = O(\varepsilon^{m-n}) \tag{2.3}$$

as the proofs of the other estimates in (2.2) are essentially similar.

Transforming the first two equations of systems (1.3) and (1.4) to the independent variables φ_{n-1} and ψ_{n-1}, which are monotone functions of time, we obtain equations in standard form:

$$d\xi/d\varphi_{n-1} = \varepsilon f(\xi, \varphi_{n-1}, \varepsilon) + \varepsilon^{m-n+2}H \tag{2.4}$$

$$d\eta/d\psi_{n-1} = \varepsilon f(\eta, \psi_{n-1}, \varepsilon) \tag{2.5}$$

where H is uniformly bounded with respect to ξ in D and ε in $[0, \varepsilon_0]$. Applying standard inequalities of the method of averaging [9] to Eqs (2.4) and (2.5), we obtain an estimate

$$|\xi(\varphi_{n-1}=s)-\eta(\psi_{n-1}=s)| = O(\varepsilon^{m-n+1}) \tag{2.6}$$

which is true for s values in an interval from $s_0=\varphi_{n-1}(t=0)=\psi_{n-1}(t=0)$ to $s_*, |s_*-s_0|=O(\varepsilon^{-1})$. Observe that the variables φ_{n-1} and ψ_{n-1} change by a quantity of the order of ε^{-1} over a length of time of the order of ε^{-n}.

In equality (2.6) is not quite similar to (2.1), as it describes the proximity of ξ and η as they vary over trajectories of the fast variables φ_{n-1} and ψ_{n-1}, respectively.

Taking quotients in the second equations of systems (1.3) and (1.4) and integrating along the φ_{n-1} and ψ_{n-1} trajectories, we obtain

$$\int_{s_0}^{\psi_{n-1}(t)} [\Phi_{n-1}(\eta, \psi_{n-1}, \varepsilon)]^{-1}\, d\psi_{n-1} = \int_{s_0}^{\varphi_{n-1}(t)} [\Phi_{n-1}(\xi, \varphi_{n-1}, \varepsilon) + \varepsilon^{m-n+1}G_{n-1}]^{-1}\, d\varphi_{n-1}$$

where t is some specific time in the interval $[0, t_*]$, $t_*=O(\varepsilon^{-n})$.

To fix our ideas, let us stipulate that the specific time t is such that $\psi_{n-1}(t)\in[s_0, \varphi_{n-1}(t)]$ (the proof is only slightly modified if this not the case). Expanding $\Phi_{n-1}(\eta, \psi_{n-1}, \varepsilon)$ in Taylor series about $\eta=\xi$ and using (2.6), we obtain

$$\int_{s_0}^{\psi_{n-1}(t)} [\Phi_{n-1}(\xi(\psi_{n-1}), \psi_{n-1}, \varepsilon) + O(\varepsilon^{m-n+1})]^{-1}\, d\psi_{n-1} =$$

$$= \int_{s_0}^{\varphi_{n-1}(t)} [\Phi_{n-1}(\xi, \varphi_{n-1}, \varepsilon) + \varepsilon^{m-n+1}G_{n-1}]^{-1}\, d\varphi_{n-1}$$

Estimating the integrals appearing in this equality over their common path of integration $[s_0, \psi_{n-1}(t)]$, (that the path of integration is indeed the same for both follows from our assumptions), and using the fact that Φ_{n-1} is bounded away from zero, we obtain

$$[\psi_{n-1}(t) - s_0] \times O(\varepsilon^{m-n+1}) = \int\limits_{\psi_{n-1}(t)}^{\varphi_{n-1}(t)} [\Phi_{n-1}(\xi, \varphi_{n-1}, \varepsilon) + \varepsilon^{m-n+1} G_{n-1}]^{-1} d\varphi_{n-1}$$

whence, since Φ_{n-1} is bounded away from zero and the intervals over which φ_{n-1} and ψ_{n-1} vary are of the order of ε^{-1}, we obtain (2.3).

Theorem 1. Under the assumptions listed in Sec. 1, the following estimates hold over time intervals of the order of ε^{-n}:

$$|x - x^\circ| = O(\varepsilon^{m-n+1})$$
$$|\alpha_i - \alpha_i^\circ| = O(\varepsilon^{m-2n+i+1}), \quad i = n-1, \ldots, 0$$

where x° and α_i° $(i = n-1, \ldots, 0)$ are approximate solutions of the original system of equations (1.1), obtained by changing the variables in accordance with (1.2) and integrating the truncated equations (1.4) instead of the exact equations (1.3) with the appropriate initial data. It must also be assumed that the η trajectory, together with a certain neighbourhood of the order of ε^{m-n+1}, lies in D.

3. SYSTEM OF EQUATIONS WITH HIDDEN HIERARCHY OF PHASE ROTATION SPEEDS

Consider the system of equations

$$x^{\cdot} = \varepsilon S_n + \varepsilon^n X \qquad (3.1)$$
$$\alpha_{n-1}^{\cdot} = \varepsilon S_{n-1} + \varepsilon^{n-1} A_{n-1}, \ \ldots, \ \alpha_1^{\cdot} = \varepsilon A_1, \quad \alpha_0^{\cdot} = A_0$$

The functions $X, A_{n-1}, \ldots, A_0$ on the right (which depend on $x, \alpha_{n-1}, \ldots, \alpha_0, \varepsilon$) have the same properties as the similarly named functions in Sec. 1; the functions $S_n, S_{n-1}, \ldots, S_2$ (also dependent on $x, \alpha_{n-1}, \ldots, \alpha_0, \varepsilon$) have the same properties as the functions $X, A_{n-1}, \ldots, A_2$ in the corresponding equations, except that they are not assumed to be bounded away from zero.

Suppose the functions S_i $(i = n, \ldots, 2)$ have the following property: if averaged over the trajectories of the fast motions (beginning with the fastest), the smallness of the result varies at each step by one order of magnitude, so that after $i-1$ averagings the final result is of order ε^{i-1}. Then there exists a sequence of n changes of variables that reduces Eqs (3.1) to an equivalent system of equations (1.3) of the mth approximation; the kth change of variable is

$$y = z + \varepsilon u_n, \ \beta_{n-1} = \gamma_{n-1} + \varepsilon u_{n-1}, \ \ldots, \ \beta_k = \gamma_k + \varepsilon u_k \qquad (3.2)$$

where $u_n, \ldots, u_k$ (which are functions of $z, \gamma_{n-1}, \ldots, \gamma_{k-1}, \varepsilon$) are uniformly bounded with respect to z in D and ε in $[0, \varepsilon_0]$.

Theorem 2. Under the above conditions the following estimates hold over a time interval of the order of ε^{-n}:

$$|x - x^\circ| = O(\varepsilon), \quad |\alpha_i - \alpha_i^\circ| = O(\varepsilon^j), \ i = n-1, \ldots, 1$$
$$j = \min\{1, m - 2n + i + 1\}, \quad |\alpha_0 - \alpha_0^\circ| = O(\varepsilon^{m-2n+1})$$

if $m < 2n-1$; but if $m \geqslant 2n-1$ one has estimates

$$|x - x^\circ| = O(\varepsilon^{m-2n+2}), \quad |\alpha_i - \alpha_i^\circ| = O(\varepsilon^{m-2n+2})$$
$$i = n-1, \ \ldots, \ 1; \quad |\alpha_0 - \alpha_0^\circ| = O(\varepsilon^{m-2n+1})$$

where x° and α_i° are the approximate solutions of Eqs (3.1) obtained by changing variables in accordance with (3.2) and integrating the truncated equations (1.4) instead of the exact system (1.3) with appropriate initial data. It must also be assumed that the η trajectory, together with a certain neighbourhood of the order of ε^j, $j = \max\{1, m-2n+2\}$, lies in D.

As an example of the application of Theorem 2 we cite the justification [10] of a sequence of averaging procedures in the resonant case in an essentially non-linear system. In that case the original formulation of the problem may be replaced by a system of equations

$$\begin{gathered} x^{\cdot} = \varepsilon S(x, \alpha_1, \alpha_0, \varepsilon) + \varepsilon^2 X(x, \alpha_1, \alpha_0, \varepsilon) \\ \alpha_1^{\cdot} = \varepsilon A_1(x, \alpha_1, \alpha_0, \varepsilon) \\ \alpha_0^{\cdot} = A_{00}(x, \alpha_1, \varepsilon) + \varepsilon A_{01}(x, \alpha_1, \alpha_0, \varepsilon) \end{gathered}$$

which satisfies the assumptions of Theorem 2 (with the sole exception that x is not a scalar variable but a vector with two components). The functions $S(x, \alpha_1, \alpha_0, \varepsilon)$ can be expanded in a Fourier series in terms of α_0 with a zero central term, and the changes of variable (3.2) are readily constructed.

The lemma and both theorems carry over without any change to the case in which the slowest variable is not a scalar but a vector of arbitrary dimensions.

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